High Dimensional Expansion

Roy Meshulam
Technion – Israel Institute of Technology

Hausdorff Research Institute for Mathematics
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Plan

Coboundary Expansion

- The $k$-dimensional Cheeger constant
- Homology of random complexes
- Expansion and topological overlap
- Expansion via symmetry
- 2-expanders from random Latin squares

Spectral Expansion

- Spectral gap of the $k$-Laplacian
- Spectral gap and colored simplices
- Garland’s method
- Homology of random flag complexes
- Spectral gap and hypergraph matching
Graphical Cheeger Constant

Edge Cuts
For a graph \( G = (V, E) \) and \( S \subset V \), \( \overline{S} = V - S \) let

\[
e(S, \overline{S}) = |\{ e \in E : |e \cap S| = 1 \}|.
\]

Cheeger Constant

\[
h(G) = \min_{0 < |S| \leq \frac{|V|}{2}} \frac{e(S, \overline{S})}{|S|}.
\]
Graphical Spectral Gap

Laplacian Matrix

$G = (V, E)$ a graph, $|V| = n$.

The Laplacian of $G$ is the $V \times V$ matrix $L_G$:

$$L_G(u, v) = \begin{cases} 
\deg(u) & u = v \\
-1 & uv \in E \\
0 & \text{otherwise}.
\end{cases}$$

Eigenvalues of $L_G$

$$0 = \lambda_1(G) \leq \lambda_2(G) \leq \cdots \leq \lambda_n(G).$$

$$\lambda_2(G) = \text{Spectral Gap of } G.$$
Cheeger Constant vs. Spectral Gap

Theorem [Alon-Milman, Tanner]:
For all $\emptyset \neq S \subseteq V$

$$ e(S, \overline{S}) \geq \frac{|S||\overline{S}|}{n} \lambda_2(G). $$

In particular

$$ h(G) \geq \frac{\lambda_2(G)}{2}. $$

Theorem [Alon, Dodziuk]:
If $G$ is $d$-regular then

$$ h(G) \leq \sqrt{2d\lambda_2(G)}. $$

$h(G)$ and $\lambda_2(G)$ are therefore essentially equivalent measures of graphical expansion.
High Dimensional Expansion

The notions of **Cheeger Constant** and **Spectral Gap** have natural high dimensional extensions. They are however not equivalent in dimensions greater than one.

**Coboundary Expansion**
- **Linial-M-Wallach**: Homology of random complexes.
- **Gromov**: The topological overlap property.
- **Gundert-Wagner**: Expansion of random complexes.

**Spectral Expansion**
- **Garland**: Cohomology of discrete groups.
- **Aharoni-Berger-M**: Hypergraph matching.
- **Kahle**: Homology of random flag complexes.
Simplicial Cohomology

$X$ a simplicial complex on $V$, $R$ a fixed abelian group.

$i$-face of $\sigma = [v_0, \cdots, v_k]$ is $\sigma_i = [v_0, \cdots, \hat{v}_i, \cdots, v_k]$.

$C^k(X) = k$-cochains = skew-symmetric maps $\phi : X(k) \to R$.

Coboundary Operator $d_k : C^k(X) \to C^{k+1}(X)$ given by

$$d_k \phi(\sigma) = \sum_{i=0}^{k+1} (-1)^i \phi(\sigma_i) .$$

$d_{-1} : C^{-1}(X) = R \to C^0(X)$ given by

$d_{-1} a(v) = a$ for $a \in R$, $v \in V$.

$Z^k(X) = k$-cocycles = ker($d_k$).

$B^k(X) = k$-coboundaries = Im($d_{k-1}$).

$k$-th reduced cohomology group of $X$:

$$\tilde{H}^k(X) = \tilde{H}^k(X; R) = Z^k(X)/B^k(X) .$$
Cut of a Cochain

Cut determined by a $k$-cochain $\phi \in C^k(X; R)$:

$$\text{supp}(d_k\phi) = \{ \tau \in X(k+1) : d_k\phi(\tau) \neq 0 \}.$$

Cut Size of $\phi$: $\|d_k\phi\| = |\text{supp}(d_k\phi)|$.

Example:

$$\|d_1\phi\| = |\{\sigma_1, \sigma_2\}| = 2$$
Cosystolic Norm of a Cochain

The Cosystolic Norm of a $k$-cochain $\phi \in C^k(X; R)$:

$$\|\lfloor\phi\rfloor\| = \min \{ |\text{supp}(\phi + d_{k-1}\psi)| : \psi \in C^{k-1}(X; R) \}.$$ 

Example: $\|\phi\| = 3$ but $\|\lfloor\phi\rfloor\| = 1$
Expansion of a Complex

Expansion of a Cochain
The expansion of $\phi \in C^k(X; R) - B^k(X; R)$ is

$$\frac{\|d_k\phi\|}{\|[\phi]\|}.$$ 

$k$-expansion Constant

$$h_k(X; R) = \min \left\{ \frac{\|d_k\phi\|}{\|[\phi]\|} : \phi \in C^k(X; R) - B^k(X; R) \right\}.$$ 

Remarks:

- $G$ graph $\Rightarrow h_0(G; \mathbb{F}_2) = h(G).$
- $h_k(X; R) > 0 \Leftrightarrow \tilde{H}^k(X; R) = 0.$
- In the sequel: $h_k(X) = h_k(X; \mathbb{F}_2).$
Expansion of a Simplex

\[ \Delta_{n-1} = \text{the } (n - 1)\text{-dimensional simplex on } V = [n]. \]

Claim [M-Wallach, Gromov]:

\[ h_{k-1}(\Delta_{n-1}) \geq \frac{n}{k + 1}. \]

Example:

\[ [n] = \bigcup_{i=0}^{k} V_i, \quad |V_i| = \frac{n}{k+1} \]

\[ \phi = 1_{V_0 \times \cdots \times V_{k-1}} \]

\[ \|[\phi]\| = \left(\frac{n}{k+1}\right)^k \]

\[ \|d_{k-1}\phi\| = \left(\frac{n}{k+1}\right)^{k+1} \]
A Model of Random Complexes

\( Y \) a simplicial complex, \( Y^{(i)} = i\)-dim skeleton of \( Y \).
\( Y(i) \) = oriented \( i\)-dim simplices of \( Y \).
\( f_i(Y) = |Y(i)| \).
\( \Delta_{n-1} = \) the \((n - 1)\)-dimensional simplex on \( V = [n] \).

\( Y_k(n, p) = \) probability space of all complexes

\[ \Delta_{n-1}^{(k-1)} \subset Y \subset \Delta_{n-1}^{(k)} \]

with probability distribution

\[ \Pr(Y) = p^{f_k(Y)}(1 - p)^{\binom{n}{k+1} - f_k(Y)} . \]
Fix $k \geq 1$ and a finite abelian group $R$.

**Theorem [Linial-M ’03, M-Wallach ’06]:**
For any function $\omega(n)$ that tends to infinity

$$\lim_{n \to \infty} \Pr \left[ Y \in Y_k(n, p) : \tilde{H}_{k-1}(Y; R) = 0 \right] = \begin{cases} 0 & p = \frac{k \log n - \omega(n)}{n} \\ 1 & p = \frac{k \log n + \omega(n)}{n} \end{cases}.$$

**The Relevance of Expansion:**
If $0 \neq [\phi] \in \tilde{H}^{k-1}(\Delta_{n-1}^{(k-1)})$ then

$$\Pr \left[ [\phi] \in \tilde{H}^{k-1}(Y; \mathbb{F}_2) \right] = (1 - p)\|d_k \phi\|$$

$$\leq (1 - p)^{\frac{n\|\phi\|}{k+1}}.$$
Weighted Expansion

$X$ - $n$-dimensional pure simplicial complex.
A probability distribution on $X(k)$:

$$w(\sigma) = \frac{|\{\eta \in X(n) : \sigma \subset \eta\}|}{\binom{n+1}{k+1} f_n(X)}.$$  

For $\phi \in C^k(X)$ let

$$\|\phi\|_w = \sum_{\{\sigma \in X(k) : \phi(\sigma) \neq 0\}} w(\sigma)$$

$$\|[\phi]\|_w = \min\{\|\phi + d_{k-1}\psi\|_w : \psi \in C^{k-1}(X)\}.$$  

Weighted $k$-th Expansion:

$$h_k(X) = \min \left\{ \frac{\|d_k\phi\|_w}{\|[\phi]\|_w} : \phi \in C^k(X) - B^k(X) \right\}.$$
The Affine Overlap Property

Number of Intersecting Simplices
For \( A = \{a_1, \ldots, a_n\} \subset \mathbb{R}^k \) and \( p \in \mathbb{R}^k \) let

\[
\gamma_A(p) = |\{\sigma \subset [n] : |\sigma| = k + 1, \ p \in \text{conv}\{a_i\}_{i \in \sigma}\}|.
\]

**Theorem [Bárány]:**
There exists \( p \in \mathbb{R}^k \) such that

\[
f_A(p) \geq \frac{1}{(k + 1)^k} \binom{n}{k + 1} - O(n^k).
\]
The Topological Overlap Property

Number of Intersecting Images

For a continuous map \( f : \Delta_{n-1} \to \mathbb{R}^k \) and \( p \in \mathbb{R}^k \) let

\[
\gamma_f(p) = |\{\sigma \in \Delta_{n-1}(k) : p \in f(\sigma)\}|.
\]

Theorem [Gromov]:

There exists \( p \in \mathbb{R}^k \) such that

\[
\gamma_f(p) \geq \frac{2k}{(k+1)!(k+1)} \binom{n}{k+1} - O(n^k).
\]
Topological Overlap and Expansion

Number of Intersecting Images
For a continuous map $f : X \to \mathbb{R}^k$ and $p \in \mathbb{R}^k$ let

$$\gamma_f(p) = |\{\sigma \in X(k) : p \in f(\sigma)\}|.$$

Expansion Condition on $X$
Suppose that for all $0 \leq i \leq k - 1$

$$h_i(X) \geq \epsilon.$$

Theorem [Gromov]
There exists a $\delta = \delta(k, \epsilon)$ such that for any continuous map $f : X \to \mathbb{R}^k$ there exists a $p \in \mathbb{R}^k$ such that

$$\gamma_f(p) \geq \delta f_k(X).$$
**Example: Symmetric Matroids**

**Matroid:**
An $n$-dimensional simplicial complex $M \subset 2^V$ such that $M[S]$ is pure for all $S \subset V$.

**Example: Linear Matroids**
$A$ - finite subset of a vector space.
$M = \text{all linearly independent subsets } \sigma \subset A$.

**Homology of matroids:**
$\tilde{H}_i(M) = 0$ for all $0 \leq i \leq \dim M - 1$.

**Symmetric matroid:**
$G = \text{Aut}(M)$ is transitive on the maximal faces.
Some Symmetric Matroids

The Partition Matroid $X_{n,m}$
Let $V_1, \ldots, V_{n+1}$ be $n + 1$ disjoint sets, $|V_i| = m$.

$$X_{n,m} = \{ \sigma \subset \bigcup_{i=1}^{n+1} V_i : \forall i \quad |\sigma \cap V_i| \leq 1 \}.$$  

Independence Matroid of Affine Space

$$\text{IN}(\mathbb{F}_q^n) = \{ \sigma \subset \mathbb{F}_q^n : \sigma \text{ is linearly independent} \}.$$  

Hermitian Unital with 65 Points

Independence matroid of the curve

$$H = \{ [x, y, z] \in PG(2, 16) : x^5 + y^5 + z^5 = 0 \}.$$
Expansion of Symmetric Matroids

Proposition [Lubotzky-M-Mozes]:

\[ M \text{ symmetric matroid} \implies h_k(M) \geq 8^{-\dim M} \quad \forall k \leq \dim M - 1. \]

Example: The Partition Matroid \( X_{n,m} \)

For \( 0 \leq k \leq n - 1 \)

\[
h_k(X_{n,m}) \geq \frac{(n+1)}{\sum_{j=0}^{k+1} \left( \frac{2(m-1)}{m} \right)^j \binom{n-j}{n-k-1}}.\]

In particular

\[
h_k(\text{Octahedral } n - \text{ sphere}) = h_k(X_{n,2}) \geq 1
\]

and

\[
h_{n-1}(X_{n,m}) \geq \frac{n+1}{\sum_{j=0}^{n} \left( \frac{2(m-1)}{m} \right)^j} > \frac{n+1}{2^{n+1} - 1}.
\]
Example: The Spherical Buildings $\Delta = A_{n+1}(\mathbb{F}_q)$

Vertices: All nontrivial linear subspaces $0 \neq V \subsetneq \mathbb{F}_q^{n+2}$.

Simplices: $V_0 \subset \cdots \subset V_k$.

Homology of $\Delta$ [Solomon, Tits]:
$\tilde{H}_i(\Delta) = 0$ for $i < n$ and $\dim \tilde{H}_n(\Delta) = q^{\binom{n+2}{2}}$.

Proposition [Gromov, LMM]:

$$h_{n-1}(A_{n+1}(\mathbb{F}_q)) \geq \frac{1}{(n+2)!}.$$ 

Problem:
For fixed $n \geq 2$ determine
$$\lim_{q \to \infty} h_{n-1}(A_{n+1}(\mathbb{F}_q)).$$
Expander Graphs

\((d, \epsilon)\)-Expanders

A family of graphs \(\{G_n = (V_n, E_n)\}_n\) with \(|V_n| \to \infty\)
with two seemingly contradicting properties:

- **High Connectivity**: \(h(G_n) \geq \epsilon\).
- **Sparsity**: \(\max_v \deg_{G_n}(v) \leq d\).

**Pinsker:**
Random \(3 \leq d\)-regular graphs are \((d, \epsilon)\)-expanders.

**Margulis:**
Explicit construction of expanders.

**Lubotzky-Phillips-Sarnak, Margulis:**
Ramanujan Graphs - an "optimal" family of expanders.
Expander Complexes

Degree of a Simplex
For $\sigma \in X(k-1)$ let $\deg(\sigma) = |\{\tau \in X(k) : \sigma \subset \tau\}|$.

$D_{k-1}(X) = \max_{\sigma \in X(k-1)} \deg(\sigma)$.

$(k, d, \epsilon)$-Expanders
A family of Complexes $\{X_n\}_n$ with $f_0(X_n) \to \infty$ such that

$D_{k-1}(X_n) \leq d$ and $h_{k-1}(X_n) \geq \epsilon$.

Problems
For fixed $k \geq 2, d, \epsilon > 0$ construct:

- $(k, d, \epsilon)$-expanders.
- Complexes that are jointly $(j, d, \epsilon)$-expanders for all $j \leq k$. 
Latin Squares

Definitions

$S_n = \text{Symmetric group on } [n]$. 
$(\pi_1, \ldots, \pi_k) \in S_n^k$ is legal if $\pi_i(\ell) \neq \pi_j(\ell)$ for all $\ell$ and $i \neq j$. 
A Latin Square is a legal $n$-tuple $L = (\pi_1, \ldots, \pi_n) \in S_n^n$. 
$L_n = \text{Latin squares of order } n$ with uniform measure.

The Usual Picture

$L = (\pi_1, \ldots, \pi_n) \leftrightarrow T_L \in M_{n \times n}([n])$

$T_L(i, \pi_k(i)) = k$ for $1 \leq i, k \leq n$.

Example for $n = 4$

$\pi = (1234)$

$L = (Id, \pi, \pi^2, \pi^3)$

$T_L =$

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The Complete 3-Partite Complex

\[ V_1 = \{a_1, \ldots, a_n\} , \quad V_2 = \{b_1, \ldots, b_n\} , \quad V_3 = \{c_1, \ldots, c_n\} \]

\[ T_n = V_1 \ast V_2 \ast V_3 = \{\sigma \subset V : |\sigma \cap V_i| \leq 1 \text{ for } 1 \leq i \leq 3\} \]

\[ T_n \cong S^2 \lor \cdots \lor S^2 \quad (n - 1)^3 \text{ times} \]
Latin Square Complexes

$L = (\pi_1, \ldots, \pi_n) \in \mathcal{L}_n$ defines a complex $Y(L) \subset T_n$ by

$$Y(L)(2) = \{ [a_i, b_j, c_{\pi_i(j)}] : 1 \leq i, j \leq n \}.$$

Example: $n = 2$

$$L = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \quad \quad \quad \quad \quad \quad Y(L) = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \quad \quad \quad \quad \quad \quad Y \left( \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \right) \cup Y \left( \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \right) = T_2$$
Random Latin Squares Complexes

Multiple Latin Squares
For $\mathcal{L}^d = (L_1, \ldots, L_d) \in \mathcal{L}_n^d$ let $Y(\mathcal{L}^d) = \bigcup_{i=1}^{d} Y(L_i)$.

The Probability Space $\mathcal{Y}(n, d)$
$\mathcal{L}_n^d = d$-tuples of Latin squares of order $n$ with uniform measure. $\mathcal{Y}(n, d) = \{ Y(\mathcal{L}^d) : \mathcal{L}^d \in \mathcal{L}_n^d \}$ with induced measure from $\mathcal{L}_n^d$.

Theorem [Lubotzky-M]:
There exist $\epsilon > 0$, $d < \infty$ such that
$$\lim_{n \to \infty} \Pr \left[ Y \in \mathcal{Y}(n, d) : h_1(Y) > \epsilon \right] = 1.$$ 

Remark: $\epsilon = 10^{-11}$ and $d = 10^{11}$ will do.
Idea of Proof

Fix $0 < c < 1$ and let $\phi \in C^1(T_n; \mathbb{F}_2)$.

\[ \phi \text{ is } \begin{cases} 
  c - \text{small} & \text{if } \|\phi\| \leq cn^2 \\
  c - \text{large} & \text{if } \|\phi\| \geq cn^2 
\end{cases} \]

**c-Small Cochains**

Lower bound on expansion in terms of the spectral gap of the vertex links.

**c-Large Cochains**

Expansion is obtained by means of a new large deviations bound for the probability space $\mathcal{L}_n$ of Latin squares.
2-Expansion and Spectral Gap

Notation
For a complex $T^{(1)}_n \subset Y \subset T_n$ let:

- $Y_v = \text{lk}(Y, v)$ = the link of $v \in V$.
- $\lambda_v$ = spectral gap of the $n \times n$ bipartite graph $Y_v$.
- $\tilde{\lambda} = \min_{v \in V} \lambda_v$.
- $d = D_1(Y)$ = maximum edge degree in $Y$.

Theorem [LM]:
If $\|[\phi]\| \leq cn^2$ then

$$\|d_1 \phi\| \geq \left(\frac{(1 - c^{1/3})\tilde{\lambda}}{2} - \frac{d}{3}\right) \|[\phi]\|.$$
Large Deviations for Latin Squares

The Random Variable $f_\mathcal{E}$

- $\mathcal{E}$ - a family of 2-simplices of $T_n$, $|\mathcal{E}| \geq cn^3$.

For a Latin square $L \in \mathcal{L}_n$ let

$$f_\mathcal{E}(L) = |Y(L) \cap \mathcal{E}|.$$

Then

$$E[f_\mathcal{E}] = \frac{|\mathcal{E}|}{n} \geq cn^2.$$

Theorem [LM]:

For all $n \geq n_0(c)$

$$\Pr[L \in \mathcal{L}_n : f_\mathcal{E}(L) < 10^{-3} c^2 n^2] < e^{-10^{-3} c^2 n^2}.$$
A positive weight function \( c(\sigma) \) on the simplices of \( X \) induces an Inner product on \( C^k(X) = C^k(X; \mathbb{R}) \):

\[
(\phi, \psi) = \sum_{\sigma \in X(k)} c(\sigma) \phi(\sigma) \psi(\sigma) .
\]

**Adjoint** \( d_k^* : C^{k+1}(X) \to C^k(X) \)

\[
(d_k \phi, \psi) = (\phi, d_k^* \psi) .
\]

\[
C^{k-1}(X) \xleftarrow{d_{k-1}} C^k(X) \xrightarrow{d_k} C^{k+1}(X)
\]

The reduced \( k \)-Laplacian of \( X \) is the positive semidefinite operator

\[
\Delta_k = d_{k-1}d_{k-1}^* + d_k^*d_k : C^k(X) \to C^k(X) .
\]
Matrix Representation of $\Delta_k$

For the constant weight function $c \equiv 1$, the matrix form of the Laplacian is

$$\Delta_k(\sigma, \tau) = \begin{cases} 
\deg(\sigma) + k + 1 & \sigma = \tau \\
(\sigma : \sigma \cap \tau) \cdot (\tau : \sigma \cap \tau) & |\sigma \cap \tau| = k, \sigma \cup \tau \notin X
\end{cases}$$

Relation with the Graph Laplacian

Let $G = 1$-skeleton of $X$

$$\Delta_0 = L_G + J$$

$$\mu_0(X) = \lambda_2(G)$$
Harmonic Cochains

The space of Harmonic $k$-cochains

$$\ker \Delta_k = \{ \phi \in C^k(X) : d_k\phi = 0, \ d^*_{k-1}\phi = 0 \}.$$ 

Simplicial Hodge Theorem:

$$C^k(X) = \text{Im} \ d_{k-1} \oplus \ker \Delta_k \oplus \text{Im} \ d^*_k.$$ 

$$\ker \Delta_k \cong \tilde{H}^k(X; \mathbb{R}).$$ 

$\mu_k(X) =$ minimal eigenvalue of $\Delta_k$. 

A Vanishing Criterion:

$$\mu_k(X) > 0 \iff \tilde{H}_k(X; \mathbb{R}) = 0.$$
Spectral Gap and Colorful Simplices

\[ \Delta_{n-1}^{(k-1)} \subset X \subset \Delta_{n-1}^{(k)} \] with vertex coloring: \([n] = V_0 \cup \cdots \cup V_k\).

Number of colorful \(k\)-simplices:

\[ e(V_0, \ldots, V_k) = |\{\sigma \in X(k) : |\sigma \cap V_i| = 1 \ \forall \ 0 \leq i \leq k\}|. \]

**Theorem [Parzanchevski-Rosenthal-Tessler]:**

Let \(c\) be the constant weight function \(c(\sigma) \equiv 1\). Then

\[ e(V_0, \ldots, V_k) \geq \frac{\prod_{i=0}^{k} |V_i|}{n} \cdot \mu_{k-1}(X). \]
Sketch of Proof

Define $\psi \in C^k(\Delta_{n-1})$ by

$$\psi([v_0, \ldots, v_k]) = \begin{cases} \text{sgn}(\pi) & v_{\pi(i)} \in V_i \quad \forall \ 0 \leq i \leq k \\ 0 & [v_0, \ldots, v_k] \text{ is not colorful.} \end{cases}$$

Let $\phi = d^*_{k-1}\psi \in C^{k-1}(\Delta_{n-1}) = C^{k-1}(X)$. Then:

$$\langle \Delta_{k-1}\phi, \phi \rangle = \langle d_{k-1}\phi, d_{k-1}\phi \rangle = n^2 \cdot e(V_0, \ldots, V_k)$$

$$\langle \phi, \phi \rangle = n \prod_{i=0}^{k} |V_i|.$$

Therefore, by the variational principle:

$$\mu_{k-1}(X) \leq \frac{\langle \Delta_{k-1}\phi, \phi \rangle}{\langle \phi, \phi \rangle} = \frac{n \cdot e(V_0, \ldots, V_k)}{\prod_{i=0}^{k} |V_i|}.$$
Eigenvalues and Cohomology

Let $X$ be a pure $d$-dimensional complex with weight function:

$$c(\sigma) = (d - \dim \sigma)!|\{\tau \in X(d) : \tau \supset \sigma\}|.$$

For $\tau \in X$ consider the link $X_\tau = \text{lk}(X, \tau)$ with a weight function given by $c_\tau(\alpha) = c(\tau \alpha)$.

**Theorem [Garland '72]:**

Let $0 \leq \ell < k < d$. Then:

$$\min_{\tau \in X(\ell)} \mu_{k-\ell-1}(X_\tau) > \frac{\ell + 1}{k + 1} \Rightarrow H^k(X; \mathbb{R}) = 0.$$

In particular:

$$\min_{\tau \in X(d-2)} \mu_0(X_\tau) > \frac{d - 1}{d} \Rightarrow H^{d-1}(X; \mathbb{R}) = 0.$$
Complexes with Expanding Links

The Projective Plane Graph

$G_q = (V_q, E_q)$: points vs. lines graph of $\text{PG}(2, q)$.

$$|V_q| = 2(q^2 + q + 1), \quad |E_q| = (q + 1)(q^2 + q + 1).$$

Spectral Gap: $\mu_0(G_q) = 1 - \frac{\sqrt{q}}{q+1}$.

If $q \geq d^2$ then $\mu_0(G_q) > \frac{d-1}{d}$. This implies the following

Theorem [Garland]:

Let $q \geq d^2$ and let $X$ be a pure $d$-dimensional complex such that $\text{lk}(X, \tau) \cong G_q$ for all $\tau \in X(d - 2)$. Then $H_{d-1}(X; \mathbb{R}) = 0$. 
Cohomology of Discrete Subgroups

$\mathbb{K}$ a local field with residue field $\mathbb{F}_q$.
$\Gamma$ a torsion-free discrete cocompact subgroup of $SL_{d+1}(\mathbb{K})$.

**Theorem [Garland]:**
If $q \geq d^2$ then $H^i(\Gamma; \mathbb{R}) = 0$ for $0 < i < d$.

**Sketch of Proof:**
$\mathcal{B} = \tilde{A}_d(\mathbb{K})$ - the affine building associated to $SL_{d+1}(\mathbb{K})$.
$\mathcal{B}$ is a contractible complex with a free $\Gamma$ action.
The quotient space $\mathcal{B}\Gamma = \mathcal{B}/\Gamma$ is a pure $d$-dimensional complex such that $\text{lk}(\mathcal{B}\Gamma, \tau) \cong G_q$ for all $\tau \in \mathcal{B}\Gamma(d-2)$.
Therefore for all $0 < i < d$

$$H^i(\Gamma; \mathbb{R}) = H^i(\mathcal{B}\Gamma; \mathbb{R}) = 0.$$
Flag Complexes

The flag complex $X(G)$ of a graph $G = (V, E)$:
Vertex set: $V$, Simplices: all cliques $\sigma$ of $G$.

Remark:
The first subdivision of a complex is a flag complex.
Face Numbers of Flag Complexes

Octahedral $n$-Sphere

$$(S^0)^{(k+1)} = \{a_1, b_1\} \ast \cdots \ast \{a_{k+1}, b_{k+1}\}$$

**Proposition [M ’03]:**

If $\tilde{H}_k(X(G)) \neq 0$ then for all $j$:

$$f_j(X(G)) \geq f_j((S^0)^{(k+1)}) = \binom{k+1}{j+1} 2^{j+1}.$$
Homology of Flag Complexes of Random Graphs

Let $\epsilon > 0$ be fixed and let $G \in G(n, p)$.

**Theorem [Kahle ’12]:**

\[
p \leq n^{\frac{1}{k} - \epsilon} \implies H_k(X(G); \mathbb{Z}) = 0 \text{ a.a.s.}
\]

\[
p \geq \left( \frac{\left( \frac{k}{2} + 1 + \epsilon \right) \log n}{n} \right)^{\frac{1}{k+1}} \implies H_k(X(G); \mathbb{R}) = 0 \text{ a.a.s.}
\]

**Theorem [DeMarco-Hamm-Kahn ’12]:**

\[
p \geq \left( \frac{\left( \frac{3}{2} + \epsilon \right) \log n}{n} \right)^{\frac{1}{2}} \implies H_1(X(G); \mathbb{F}_2) = 0 \text{ a.a.s.}
\]
Eigenvalues of Flag Complexes

$G = (V, E)$ graph, $|V| = n$, $X = X(G)$ with weights $c(\sigma) \equiv 1$. 
$\mu_k = \mu_k(X) =$ minimal eigenvalue of $\Delta_k$ on $X$.

**Theorem [Aharoni-Berger-M]:**

For $k \geq 1$

$$k \mu_k \geq (k + 1) \mu_{k-1} - n.$$

In particular:

$$\mu_k \geq (k + 1) \lambda_2 - kn.$$

**Corollary:**

$$\lambda_2(G) > \frac{kn}{k + 1} \Rightarrow \mu_k > 0 \Rightarrow \tilde{H}^k(X(G)) = 0.$$
Example: Turán Graph

\[ |V_1| = \cdots = |V_k| = \ell, \ n = k\ell, \ m = (\ell - 1)^k. \]

\( T_k(n) \) - the complete \( k \)-partite graph on \( V_1 \cup \cdots \cup V_k \).

**Spectral gap**

\[ \lambda_2(T_k(n)) = \frac{(k - 1)n}{k}. \]

**Flag complex**

\[ X(T_k(n)) = V_1 \ast \cdots \ast V_k \simeq \bigvee_{i=1}^{m} S^{k-1}. \]

\[ \dim \tilde{H}_{k-1}(X(T_k(n)); \mathbb{R}) = m \neq 0. \]
Eigenvalues and Connectivity of $I(G)$

The independence complex $I(G)$

Vertex set: $V$, Simplices: all independent sets $\sigma$ of $G$.

Homological connectivity

$$\eta(Y) = 1 + \min\{i : \tilde{H}_i(Y) \neq 0\}.$$ 

Theorem [ABM]:

For a graph $G$ on $n$ vertices

$$\eta(I(G)) \geq \frac{n}{\lambda_n(G)}.$$
Bipartite Matching

$A_1, \ldots, A_m$ finite sets.

A System of Distinct Representatives (SDR):

a choice of distinct $x_1 \in A_1, \ldots, x_m \in A_m$.

\[
\begin{array}{c|c|c}
A_1 & A_2 & A_3 \\
1 & 1 & 2 \\
3 & 3 & 3 \\
\end{array}
\]

$\exists$ SDR

\[
\begin{array}{c|c|c}
A_1 & A_2 & A_3 \\
1 & 1 & 2 \\
2 & 1 & 2 \\
\end{array}
\]

$\nexists$ SDR

Hall’s Theorem (1935)

$(A_1, \ldots, A_m)$ has an SDR iff

$|\bigcup_{i \in I} A_i| \geq |I|$ for all $I \subseteq [m] = \{1, \ldots, m\}$. 
Hypergraph Matching

A Hypergraph is a family of sets $\mathcal{F} \subset 2^V$
$(\mathcal{F}_1, \ldots, \mathcal{F}_m)$ a sequence of $m$ hypergraphs
A System of Disjoint Representatives (SDR) for $(\mathcal{F}_1, \ldots, \mathcal{F}_m)$
is a choice of pairwise disjoint $F_1 \in \mathcal{F}_1, \ldots, F_m \in \mathcal{F}_m$

When do $(\mathcal{F}_1, \ldots, \mathcal{F}_m)$ have an SDR?

The problem is NP-Complete even if all $\mathcal{F}_i$’s consist of 2-element sets. Therefore, we cannot expect a ”good” characterization as in Hall’s Theorem.

There are however some interesting sufficient conditions ...
Do \((F_1, F_2, F_3, F_4)\) have an SDR?
The Aharoni-Haxell Theorem

A Matching is a hypergraph $\mathcal{M}$ of pairwise disjoint sets. The Matching Number $\nu(\mathcal{F})$ of a hypergraph $\mathcal{F}$ is the maximal size $|\mathcal{M}|$ of a matching $\mathcal{M} \subset \mathcal{F}$.

$$\nu(\mathcal{F}) = 3$$

$$\nu(\mathcal{F}) = 1$$

The Aharoni-Haxell Theorem

$\mathcal{F}_1, \ldots, \mathcal{F}_m \subset \binom{\mathcal{V}}{r}$ such that for all $I \subset [m]$

$$\nu\left(\bigcup_{i \in I} \mathcal{F}_i\right) > r(|I| - 1).$$

Then $(\mathcal{F}_1, \ldots, \mathcal{F}_m)$ has an SDR.
A Fractional Extension

A Fractional Matching of a hypergraph $\mathcal{F}$ on $V$ is a function $f : \mathcal{F} \to \mathbb{R}_+$ such that $\sum_{F \ni v} f(F) \leq 1$ for all $v \in V$. The Fractional Matching Number $\nu^*(\mathcal{F})$ is

$$\max_f \sum_{F \in \mathcal{F}} f(F)$$

over all fractional matchings $f$.

Example: The Finite Projective Plane $\mathcal{P}_n$

$\nu(\mathcal{P}_n) = 1$, $\nu^*(\mathcal{P}_n) = \frac{n^2+n+1}{n+1}$

Theorem [Aharoni-Berger-M]:

$\mathcal{F}_1, \ldots, \mathcal{F}_m \subset \binom{V}{r}$ such that for all $I \subset [m]$

$$\nu^*(\bigcup_{i \in I} \mathcal{F}_i) > r(|I| - 1) .$$

Then $(\mathcal{F}_1, \ldots, \mathcal{F}_m)$ has an SDR.