

Cyclotomic quiver Hecke algebras IV

Applications and other types

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 November 2017

Ariki-Brundan-Kleshchev categorification theorem

Let C be a generalised Cartan matrix of type $A_e^{(1)}$ or A_∞ :



Last lecture we saw that

Theorem (Ariki, Brundan-Kleshchev, Brundan-Stroppel, Rouquier)

Let C be a Cartan matrix of type $A_e^{(1)}$ or A_∞ and let \mathbb{k} be a field. Then

$$L_A(\Lambda) \cong \bigoplus_{n \geq 0} \text{Proj}(\mathcal{R}_n^\Lambda) \quad \text{and} \quad L_A(\Lambda)^\vee \cong \bigoplus_{n \geq 0} \text{Rep}(\mathcal{R}_n^\Lambda)$$

Moreover, if $\mathbb{k} = \mathbb{C}$ then

- The canonical basis of $L_A(\Lambda)$ is $\{[Y^\mu] \mid \mu \in \mathcal{K}^\Lambda\}$
- The dual canonical basis of $L_A(\Lambda)$ is $\{[D^\mu] \mid \mu \in \mathcal{K}^\Lambda\}$

If $\mu \in \mathcal{K}^\Lambda$ then $(D^\mu)^\circledast \cong D^\mu$ and $(Y^\mu)^\# \cong Y^\mu$, where if M is an \mathcal{R}_n^Λ -module then $M^\circledast = \text{Hom}_{\mathbb{k}}(M, \mathbb{k})$ and $M^\# = \text{Hom}_{\mathcal{R}_n^\Lambda}(M, \mathcal{R}_n^\Lambda)$

Outline of lectures

- 1 Quiver Hecke algebras and categorification
 - Basis theorems for quiver Hecke algebras
 - Categorification of $U_q(\mathfrak{g})$
 - Categorification of highest weight modules
- 2 The Brundan-Kleshchev graded isomorphism theorem
 - Seminormal forms and semisimple KLR algebras
 - Lifting idempotents
 - Cellular algebras
- 3 The Ariki-Brundan-Kleshchev categorification theorem
 - Dual cell modules
 - Graded induction and restriction
 - The categorification theorem
- 4 Recent developments
 - Consequences of the categorification theorem
 - Webster diagrams and tableaux
 - Content systems and seminormal forms

Categorification of the canonical basis of $U_A(\widehat{\mathfrak{sl}}_e)^+$

Set $\text{Proj}(\mathcal{R}) = \bigoplus_{n \geq 0} \text{Proj}(\mathcal{R}_n)$ and let $\#$ be the automorphism of $\text{Proj}(\mathcal{R})$ induced by $M^\# = \text{Hom}_{\mathcal{R}_n}(M, \mathcal{R}_n)$

Theorem (Brundan-Kleshchev, Brundan-Stroppel, Rouquier)

Let C be a Cartan matrix of type $A_e^{(1)}$ or A_∞ and let $\mathbb{k} = \mathbb{C}$. Then $U_A^-(\widehat{\mathfrak{sl}}_e) \cong \text{Proj}(\mathcal{R})$ and the canonical basis of $U_A(\widehat{\mathfrak{sl}}_e)$ coincides with the basis of $\text{Proj}(\mathcal{R})$ of $\#$ -self-dual projective indecomposable \mathcal{R}_n -modules

Proof Let \mathbf{B} be the canonical basis of $U_A^-(\widehat{\mathfrak{sl}}_e)$ and \mathbf{B}_Λ be the canonical basis of $L_A(\Lambda) = U_A(\widehat{\mathfrak{sl}}_e)v_\Lambda$, for $\Lambda \in P^+$. Then \mathbf{B} is the unique weight basis of $U_A^-(\widehat{\mathfrak{sl}}_e)$ such that if $b \in \mathbf{B}$ then $bv_\Lambda \in \mathbf{B}_\Lambda \cup \{0\}$

As $\mathbf{B}_\Lambda = \{[Y^\mu] \mid \mu \in \mathcal{K}_n^\Lambda\}$, it is enough to show that if Y is a self-dual \mathcal{R}_n -module then $[Y]v_\Lambda$ is either zero or equal to $[Y^\mu]$, for some $\mu \in \mathcal{K}^\Lambda$

Define a functor $\text{pr}_\Lambda : \mathcal{R}_n\text{-Mod} \rightarrow \mathcal{R}_n^\Lambda\text{-Mod}$ by $\text{pr}_\Lambda M = \mathcal{R}_n^\Lambda \otimes_{\mathcal{R}_n} M$

$$\implies \text{pr}_\Lambda \text{ sends projectives to projectives and } \text{pr}_\Lambda \circ \# \cong \# \circ \text{pr}_\Lambda$$

This implies the result

Simple modules

By definition, $\mathcal{K}_n^\Lambda = \{\mu \in \mathcal{P}_n^\Lambda \mid D^\mu \neq 0\}$ but we did not describe this set

Given i -nodes $A < C$, for $i \in I$, define

$$d_A^C(\mu) = \#\{B \in \text{Add}_i(\mu) \mid A < B < C\} - \#\{B \in \text{Rem}_i(\mu) \mid A < B < C\}$$

A removable i -node A is **normal** if $d_A(\mu) \leq 0$ and $d_A^C(\mu) < 0$ whenever $C \in \text{Rem}_i(\mu)$ and $A < C$.

A normal i -node A is **good** if $A \leq B$ whenever B is a normal i -node.

Write $\lambda \xrightarrow{i\text{-good}} \mu$ if $\mu = \lambda + A$ for some good i -node A .

Misra and Miwa showed that the crystal graph of $L_\Lambda(\Lambda)$, considered as a submodule $\mathcal{F}_\Lambda^\Lambda$, is the graph with vertex set

$$\mathcal{L}_0^\Lambda = \{\mu \in \mathcal{P}_n^\Lambda \mid \mu = 0_\ell \text{ or } \lambda \xrightarrow{i\text{-good}} \mu \text{ for some } \lambda \in \mathcal{L}_0^\Lambda\},$$

and labelled edges $\lambda \xrightarrow{i\text{-good}} \mu$, for $i \in I$

Corollary (Ariki)

Suppose that \mathbb{k} is an arbitrary field and that $\mu \in \mathcal{P}_n^\Lambda$. Then $\mathcal{K}^\Lambda = \mathcal{L}_0^\Lambda$. That is, if $\mu \in \mathcal{P}_n^\Lambda$ then $D_{\mathbb{k}}^\mu \neq 0$ if and only if $\mu \in \mathcal{L}_0^\Lambda$

Proof Immediate because $[D^\mu] = [S^\mu] + \text{lower terms}$, for $\mu \in \mathcal{K}^\Lambda$

Almost simple modules

The quiver Hecke algebra $\mathcal{R}_n^\Lambda(\mathbb{Z})$ is defined over \mathbb{Z} (but $\mathcal{R}_n^\Lambda(\mathbb{Z}) \neq \mathcal{H}_n^\Lambda(\mathbb{Z})!$)

$\mathcal{R}_n^\Lambda(\mathbb{Z})$ is a \mathbb{Z} -graded \mathbb{Z} -free cellular algebra

$\implies S_{\mathbb{Z}}^\lambda$ is defined over \mathbb{Z} with a \mathbb{Z} -valued bilinear form $\langle \cdot, \cdot \rangle$

Define $\text{rad } S_{\mathbb{Z}}^\lambda = \{x \in S_{\mathbb{Z}}^\lambda \mid \langle x, y \rangle = 0 \text{ for all } y \in S_{\mathbb{Z}}^\lambda\}$

$\implies \text{rad } S_{\mathbb{Z}}^\lambda$ is a \mathbb{Z} -graded \mathbb{Z} -free submodule of $S_{\mathbb{Z}}^\lambda$

Definition

Let $E_{\mathbb{Z}}^\mu = S_{\mathbb{Z}}^\mu / \text{rad } S_{\mathbb{Z}}^\mu$. For a field \mathbb{k} , let $E_{\mathbb{k}}^\mu = E_{\mathbb{Z}}^\mu \otimes_{\mathbb{Z}} \mathbb{k}$, an $\mathcal{R}_n^\Lambda(\mathbb{k})$ -module

Theorem (M., Brundan-Kleshchev)

- 1 The module $E_{\mathbb{Z}}^\mu$ is a \mathbb{Z} -graded \mathbb{Z} -free $\mathcal{R}_n^\Lambda(\mathbb{Z})$ -module
- 2 If $\mathbb{k} = \mathbb{Q}$ then $E_{\mathbb{Q}}^\mu$ is self-dual and, moreover, $E_{\mathbb{Q}}^\mu \cong D_{\mathbb{Q}}^\mu$ is an absolutely irreducible graded $\mathcal{R}_n^\Lambda(\mathbb{Q})$ -module

- 3 For any $\lambda \in \mathcal{P}_n^\Lambda$ and $\mu \in \mathcal{K}_n^\Lambda$,

$$[S_{\mathbb{k}}^\lambda : D_{\mathbb{k}}^\mu]_q = \sum_{\nu} [S_{\mathbb{Q}}^\lambda : E_{\mathbb{Q}}^\nu]_q [E_{\mathbb{k}}^\nu : D_{\mathbb{k}}^\mu]_q = \sum_{\nu} d_{\lambda\nu}(q) [E_{\mathbb{k}}^\nu : D_{\mathbb{k}}^\mu]_q$$

adjustment matrix

Categorification of highest weight modules

The categorification of $L(\Lambda)^\vee$ and $L(\Lambda)$ by the algebras \mathcal{R}_n^Λ is extensive:

- Multiplication by q corresponds to the grading shift functor
- $E_i \leftrightarrow i\text{-Res}$ and $F_i \leftrightarrow q i\text{-Ind } K_i^{-1}$
- The weight spaces of $L(\Lambda)$ are the blocks of \mathcal{R}_n^Λ
- The Shapovalov form on $L(\Lambda)$ is the Cartan pairing on $\text{Rep}(\mathcal{R}_n^\Lambda)$
- The standard basis of $L(\Lambda)$ corresponds to the graded Specht modules
- The costandard basis of $L(\Lambda)$ corresponds to the dual graded Specht modules
- The vertices of the crystal graph label the simple modules
- The crystal graph gives the modular branching rules
- The action of the affine Weyl group corresponds to the derived equivalences of Chuang and Rouquier
- If $F = \mathbb{C}$ the dual canonical basis is the basis of irreducible modules
- If $F = \mathbb{C}$ the canonical basis is the basis of projective indecomposable modules

The James conjecture

James conjecture (1990)

Let \mathbb{k} be a field of characteristic p and $\alpha \in \mathbb{Q}^+$ such that $\text{def } \alpha < p$. Then the **adjustment matrix** for $\mathcal{R}_\alpha^{\Lambda_0} \cong \mathcal{H}_\xi(\mathfrak{S}_n)_\alpha$ is the identity matrix.

Proving the James and Lusztig conjectures motivated developments in representation theory for the last twenty years.

Evidence for James and Lusztig conjectures

- (Andersen-Jantzen-Soergel) True for almost all primes
- True for $n \leq 30$ (James, M., ...)
- True for blocks of defect/weight 1, 2 (Richards) and 3 and 4 (Fayers)
- True for the **Rouquier Blocks**, which have arbitrary weight (Chuang-Tan, James-Lyle-M.)

Williamson (2013)

The James and Lusztig conjectures are both wrong!!!

The smallest known counter-example to the James conjecture occurs in a block of defect **561** in characteristic **839** for the symmetric group $\mathfrak{S}_{467,874}$

Loadings and Webster tableaux

Recall that $\lambda \in P^+$ is a dominant weight of level ℓ .

A **loading** is a sequence $\theta = (\theta_1, \dots, \theta_\ell) \in \mathbb{Z}^\ell$ such that

$$\theta_1 < \theta_2 < \dots < \theta_\ell \text{ and } \theta_k \not\equiv \theta_l \pmod{\ell} \text{ for } 1 \leq k < l \leq \ell$$

Extend θ to the set of nodes by defining

$$\theta(l, r, c) = N\theta_l + L(c - r) + r + c - 1$$

where $L = N\ell$ and $N \gg (2n - 1)$ — different nodes have different loadings

The **loading** of $\lambda \in \mathcal{P}_n^\Lambda$ is $L_\theta(\lambda) = \{\theta(\alpha) \mid \alpha \in [\lambda]\}$

Define the **θ -dominance order** on \mathcal{P}_n^Λ by $\lambda \triangleright_\theta \mu$ if for all nodes (l, r, c)

$$\#\{\alpha \in [\lambda] \mid \theta(\alpha) \geq \theta(l, r, c)\} \geq \#\{\alpha \in [\mu] \mid \theta(\alpha) \geq \theta(l, r, c)\}$$

A **Webster λ -tableau** of type μ is a bijection $T: [\lambda] \rightarrow L_\theta(\mu)$ such that

- 1 If $1 \leq k \leq \ell$ and $\lambda^{(k)} \neq (0)$ then $T(k, 1, 1) \leq N\theta_k$
- 2 If $(k, r - 1, c), (k, r, c) \in \lambda$ then $T(k, r - 1, c) < T(k, r, c) + L$
- 3 If $(k, r, c - 1), (k, r, c) \in \lambda$ then $T(k, r, c - 1) < T(k, r, c) - L$

Let $SStd_\theta(\lambda, \mu)$ be the set of Webster λ -tableau of type μ and

let $SStd_\theta(\lambda) = \bigcup_\mu SStd_\theta(\lambda, \mu)$. Let $\omega_n = (0 \mid \dots \mid 0 \mid 1^n)$.

Then $Std_\theta(\lambda) = SStd_\theta(\lambda, \omega_n)$ is the set of **standard Webster tableau**

Graded decomposition example

Example (Bowman) Take $e = 2, n = 2, \Lambda = \Lambda_0 + \Lambda_1$ and $\kappa = (0, 1)$. Then \mathcal{R}_2^Λ has two one dimensional self-dual simple modules, $D(01)$ and $D(10)$, such that 1_i acts as δ_{ij} on $D(j)$

$$\theta = (0, 1)$$

	$D(10)$	$D(01)$
$(1^2 \mid 0)$	1	
$(0 \mid 1^2)$.	1
$(1 \mid 1)$	q	q
$(2 \mid 0)$	q^2	.
$(0 \mid 2)$.	q^2

$$\theta = (0, 3)$$

	$D(10)$	$D(01)$
$(0 \mid 1^2)$	1	
$(0 \mid 2)$	q	
$(1 \mid 1)$	q^2	1
$(1^2 \mid 0)$.	q
$(2 \mid 0)$.	q^2

Before we can define the $\{c_{st}^\theta\}$ basis we need to introduce a new algebra

Many cellular bases

Theorem (Bowman, Webster, 2017)

Let θ be a loading. Then \mathcal{R}_n^Λ has a graded cellular basis

$$\{c_{st}^\theta \mid s, t \in Std_\theta(\lambda), \lambda \in \mathcal{P}_n^\Lambda\}$$

with respect to the poset $(\mathcal{P}_n^\Lambda, \triangleright_\theta)$

Let C_λ^θ be the cell module indexed by $\lambda \in \mathcal{P}_n^\Lambda$ determined by the θ -cellular basis $\{c_{st}^\theta\}$ and let $D_\mu^\theta = C_\mu^\theta / \text{rad } C_\mu^\theta$. Define

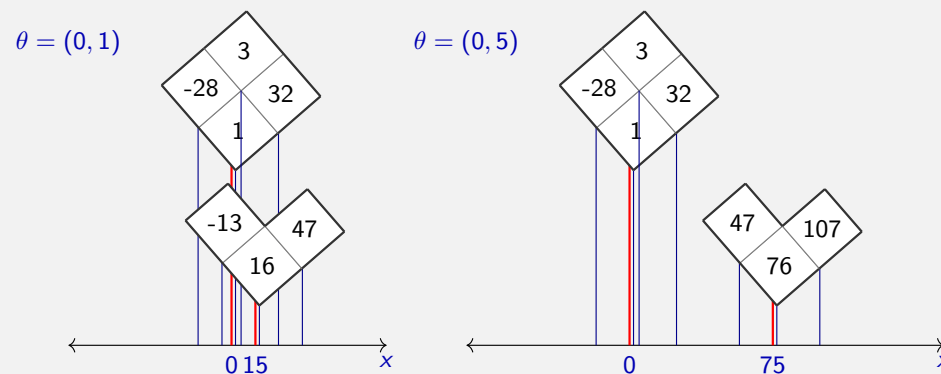
$$d_{\lambda\mu}^\theta(q) = [C_\lambda^\theta : D_\mu^\theta]_q = \sum_{k \in \mathbb{Z}} [C_\lambda^\theta : D_\mu^\theta(k)] q^k$$

The θ -cellular bases genuinely depend on θ and they are in general different from the ψ and ψ' -bases. In fact, the graded dimensions of the θ -cell modules and the graded decomposition numbers depend on θ

Webster tableau

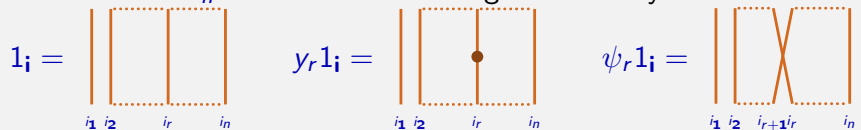
For any λ there is always a unique Webster λ -tableau of type λ . Rather than drawing this “normally” we want to draw Webster tableau using the **Russian notation**

Example Take $\lambda = (2^2 \mid 2, 1)$, so that $N = 15$ and $L = 30$, for the two loadings $\theta = (0, 1)$ and $\theta = (0, 5)$, respectively:



Webster diagrams

The elements of \mathcal{R}_n can be described diagrammatically:



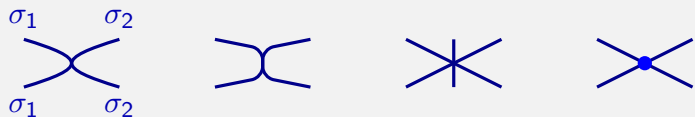
We want similar, but more complicated diagrams, to define an algebra \mathcal{W}_n^Λ

Webster diagrams have three types of strings:

- Thick red vertical strings with x -coordinates $N\theta_1, \dots, N\theta_\ell$
- Solid strings of residues i_1, \dots, i_n , for some $\mathbf{i} \in I^n$
- Dashed grey ghost strings that are translates, L -units to the left, of the solid strings. A ghost string has the same residue as the corresponding solid string

Diagrams are defined up to isotopy and solid strings can have dots

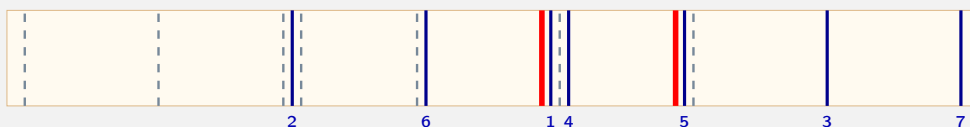
The following crossings are **not** allowed for red, solid or ghost strings):



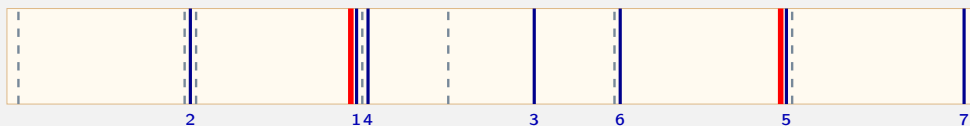
Examples of Webster diagrams II

Now let $\lambda = (2^2|2, 1)$, so that $N = 15$ and $L = 30$.

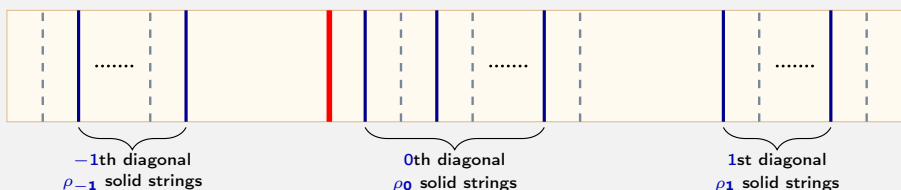
If $\theta = (0, 1)$ then $1_\lambda^{\mathbf{i}}$ is the diagram



If $\theta = (0, 5)$ then $1_\lambda^{\mathbf{i}}$ looks like:

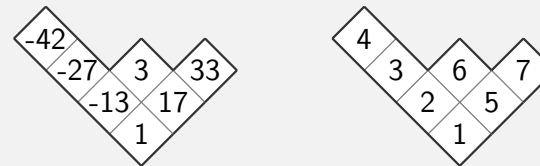


Strings in diagrams from ℓ -partitions “cluster” according to the diagonals:

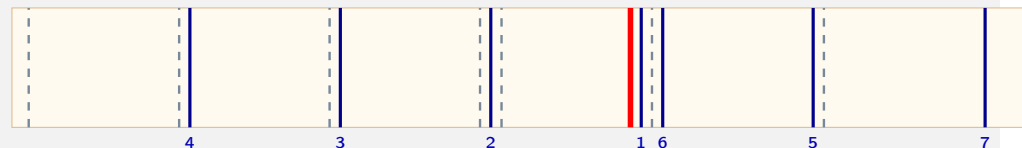


Examples of Webster diagrams

Example Let $\ell = 1$, $\theta = (0)$ and $\lambda = (4, 2, 1)$. Then $N = 15 = L$ and $SStd_\theta(\lambda, \lambda)$ contains the tableau:



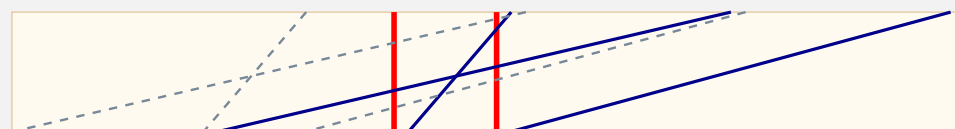
The corresponding Webster diagram 1_λ is:



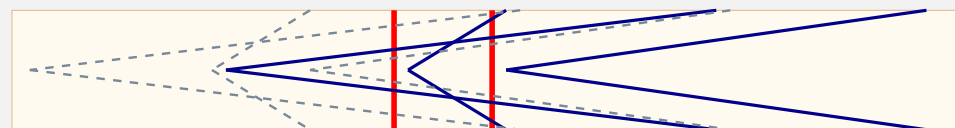
Composing Webster diagrams

We compose Webster diagrams in the usual way: if D and E are Webster diagrams then the diagram $D \circ E$ is 0 if their residues are different and when their residues are the same we put D on top of E and apply isotopy.

For example if D is the diagram

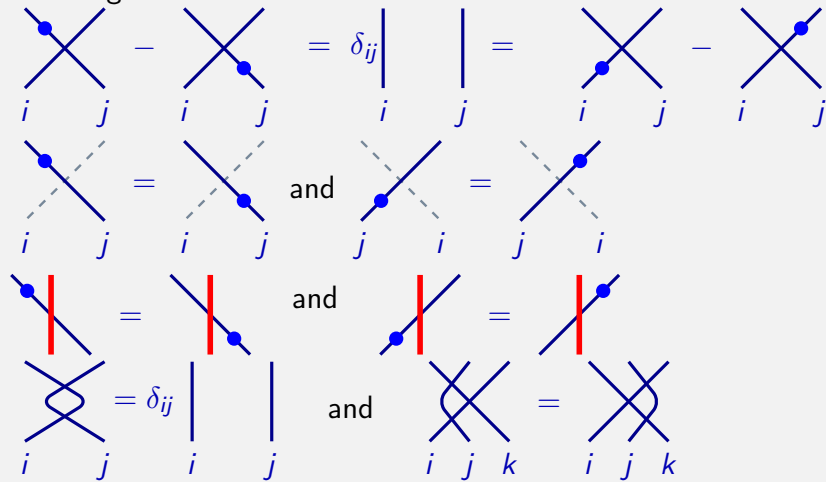


Let E be the diagram obtained by reflecting D in the line $y = 0$. Then $D \circ E$ is the diagram

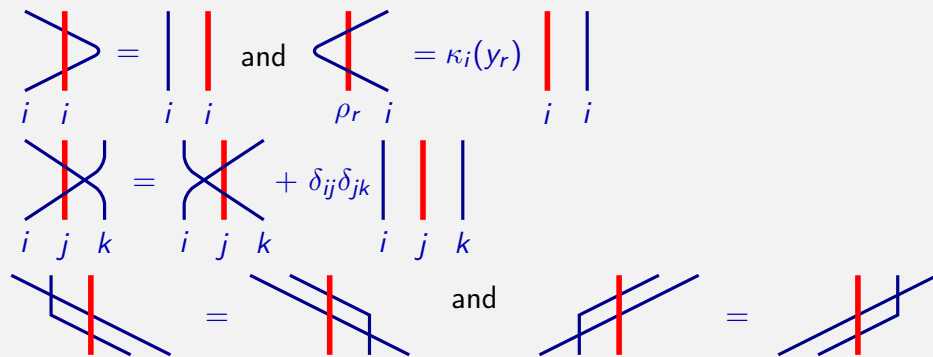


Relations for Webster algebras

The Webster algebra \mathscr{W}_n^Λ is the \mathbb{k} -algebra spanned by isotopy classes of Webster diagrams with multiplication given composition and subject to the following local relations:



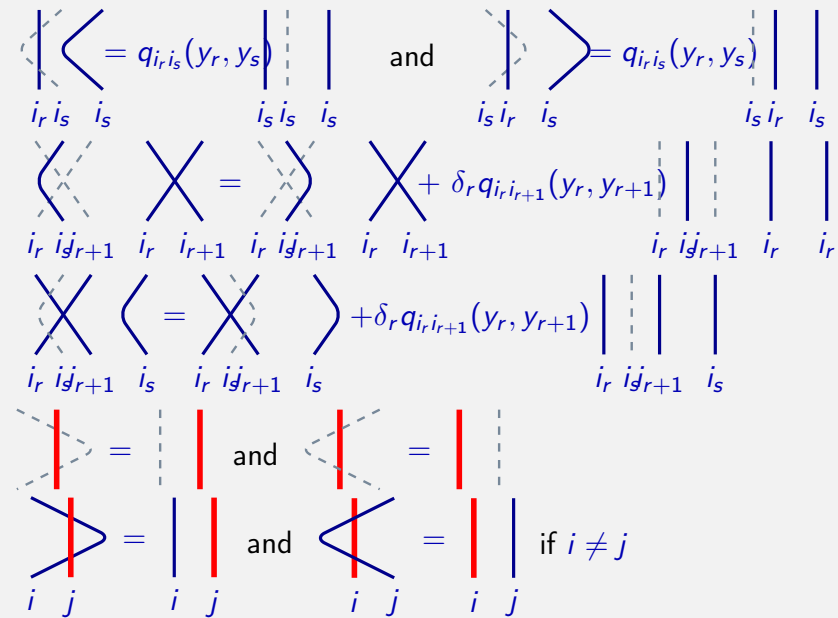
Relations for Webster algebras III



... and a solid strand in D is **unsteady** if it intersects the region $(\infty, LN] \times [0, L]$, in which case $D = 0$

These relations are homogeneous, so \mathscr{W}_n^Λ is a graded algebra

Relations for Webster algebras II



The cellular basis

Inside \mathscr{W}_n^Λ , for $T \in SStd_\theta(\lambda, \mu)$ define the diagram C_T to be a Webster diagram with a minimal number of crossings such that for each node $(l, r, c) \in [\lambda]$ there is a solid string of residue $\kappa_l + c - r + e\mathbb{Z}$ that starts with x -coordinate $T(l, r, c) \in L_\theta(\mu)$ at the top of the diagram and that finishes with x -coordinate $\theta(l, r, c) \in L_\theta(\lambda)$ at the bottom of the diagram.

The diagram C_T is not unique, in general.

Let C_T^* be the diagram obtained from C_T by reflecting it in the line $y = 0$.

Define $C_{ST}^\theta = C_S C_T^*$

Theorem (Bowman, Webster)

The algebra \mathscr{W}_n^Λ is spanned by the diagrams $\{C_{ST}^\theta \mid S, T \in SStd_\theta(\lambda) \text{ for } \lambda \in \mathcal{P}_n^\Lambda\}$

Idea of proof First push all strings to the left so that they are concave, turning at the equator. This shows that if D is a Webster diagram then $D \in \mathscr{W}_n^\Lambda 1_\lambda \mathscr{W}_n^\Lambda$, for some $\lambda \in \mathcal{P}_n^\Lambda$.

By resolving crossing it now follows that \mathscr{W}_n^Λ is spanned by the $\{C_{ST}^\theta\}$

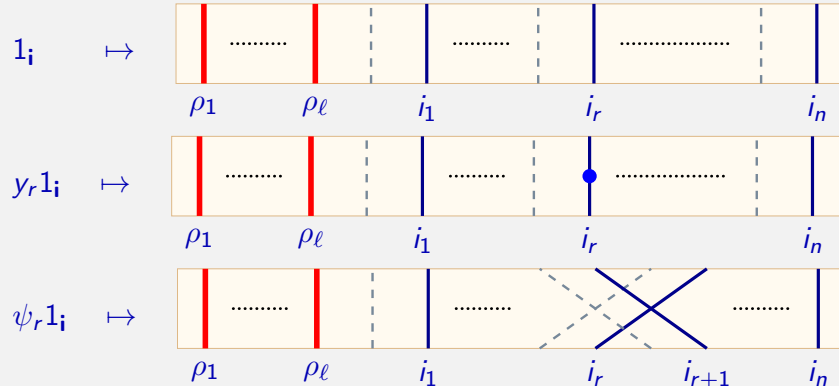
Connection to KLR

Recall that $\omega_n = (0 | \dots | 0 | 1^n)$ and that $\text{Std}_\theta(\lambda) = \text{SStd}_\theta(\lambda, \omega_n)$

Theorem (Bowman, Webster)

There is an isomorphism of graded algebras $\mathcal{R}_n^\Lambda \xrightarrow{\cong} 1_{\omega_n} \mathcal{W}_n^\Lambda 1_{\omega_n}$

Idea of proof The isomorphism is given by:



Pull all strings to the right and check the relations

Content systems

In the most general set up, the cyclotomic quiver Hecke algebra depends on choice of polynomials $\mathbf{Q}_I = (Q_{ij}(u, v))$ and $\mathbf{K}_I = (\kappa_i(u))$, so write

$$\mathcal{R}_n^\Lambda = \mathcal{R}_n^\Lambda(\mathbf{Q}_I, \mathbf{K}_I)$$

Fix $\rho \in I^\ell$ such that $\Lambda = \sum_I \Lambda_{\rho_i}$

Let Γ_ℓ be the quiver of type $A_\infty \times \dots \times A_\infty$, with ℓ factors.

More explicitly, Γ_ℓ has vertex set $J_\ell = [1, \ell] \times \mathbb{Z}$ and edges

$$(l, a) \rightarrow (l, a + 1), \text{ for all } (l, a) \in J_\ell$$

Definition (Evseev-M.)

A **content system** for $\mathcal{R}_n^\Lambda(\mathbf{Q}_I, \mathbf{K}_I)$ is a pair of maps

$$r: J_\ell \rightarrow I \text{ and } c: J_\ell \rightarrow \mathbb{k} \text{ such that}$$

- $r(l, 0) = \rho_l$ and $\kappa_i(u) = \prod_{l \in [1, \ell], \rho_l \equiv i} (u - c(l, 0))$
- If $i = r(k, a)$ then $Q_{ij}(c(k, a), v) \simeq \prod_b (v - c(k, b))$, where $b = a \pm 1$ and $r(k, b) = j$
- If $(k, a), (l, b) \in J_\ell$ then $r(k, a) = r(l, b)$ and $c(k, a) = c(l, b)$ if and only if $(k, a) = (l, b)$
- Plus one more technical constraint

The loaded cellular basis

For $s, t \in \text{Std}_\theta(\lambda)$ define c_{st}^θ to be the element of \mathcal{R}_n^Λ that is sent to C_{st}^θ under the previous isomorphism.

Corollary (Bowman, Webster)

The elements $\{c_{st}^\theta \mid (s, t) \in \text{Std}^2(\mathcal{P}_n^\Lambda)\}$ span \mathcal{R}_n^Λ

Theorem (Bowman, Webster)

The algebra \mathcal{W}_n^Λ is quasi-heredity over \mathbb{k} with graded cellular basis $\{C_{ST}^\theta \mid S, T \in \text{SStd}_\theta(\lambda) \text{ for } \lambda \in \mathcal{P}_n^\Lambda\}$

Bowman calls \mathcal{W}_n^Λ a **diagrammatic Cherednik algebra**. These algebras include, as a special case, the quiver Schur algebras of type A introduced by Stroppel and Webster (and Hu and Mathas in type A_∞). Webster proves that \mathcal{W}_n^Λ categorifies Uglov's generalised Fock spaces

The last theorem provides us with a quotient functor, or **Schur functor**:

$$E_{\omega_n}: \mathcal{W}_n^\Lambda\text{-Mod} \rightarrow \mathcal{R}_n^\Lambda\text{-Mod}; M \mapsto 1_{\omega_n} M$$

Examples of content systems

- If $\Gamma = A_\infty \sqcup \dots \sqcup A_\infty$, so that $I = J_\ell$, then $r(k, a) = (k, a)$ and $c(k, a) = 0$ is a content system with coefficients in \mathbb{Z}
- If Γ is a quiver of type $A_{e+1}^{(1)}$ then a content system is given by:

$$\begin{array}{cccccccc} r & 0 & 1 & 2 & \dots & e & 0 & 1 & \dots \\ c & 0 & x & 2x & \dots & ex & (e+1)x & (e+2)x & \dots \end{array}$$

- If Γ is a quiver of type $C_e^{(1)}$ then
- $$\begin{array}{cccccccccccc} r & 0 & 1 & \dots & e-1 & e & e-1 & \dots & 1 & 0 & 1 & \dots \\ c & 0 & x & \dots & (e-1)x & (ex)^2 & -(e+1)x & \dots & -(2e-1)x & (2x)^2 & (2e+1)x & \dots \end{array}$$

Content systems are not unique – the most generic content systems are defined over $\mathbb{Z}[x, x_1, \dots, x_\ell]$

All of these content systems, and hence the algebras $\mathcal{R}_n^\Lambda(\mathbf{Q}_I, \mathbf{K}_I)$ are defined over $\mathbb{Z}[x]$. There is a natural (homogeneous) specialisation map $\mathcal{R}_n^\Lambda(\mathbf{Q}_I, \mathbf{K}_I) \rightarrow \mathcal{R}_n^\Lambda$ given by tensoring with $\mathbb{Z}[x]/x\mathbb{Z}[x]$

Proposition

Let $\lambda \in \mathcal{P}_n^\Lambda$ and let V be the \mathbb{K} -vector space with a basis $\{v_t \mid t \in \text{Std}(\lambda)\}$ and set $v_s = 0$ if s is not standard.

Suppose that there exist scalars

$$\{\beta_r(t) \in \mathbb{K} \mid 1 \leq r < n \text{ and } t, s_r t \in \text{Std}(\lambda)\}$$

satisfying certain technical conditions.

Then V has the structure of an irreducible $\mathcal{R}_n^\Lambda(\mathbf{Q}_I, \mathbf{K}_I)$ -module where the $\mathcal{R}_n^\Lambda(\mathbf{Q}_I, \mathbf{K}_I)$ -action is determined by:

$$1_i v_t = \delta_{i,t} v_t$$

$$y_r v_t = c_r(t) v_t$$

$$\psi_r v_t = \beta_r(t) v_{s_r t} + \frac{\delta_{i_r^t, r+1}}{c_{r+1}(t) - c_r(t)} v_t$$

for all $i \in I^n$, all admissible r and all $t \in \text{Std}(\lambda)$.

Idea of Proof Check the relations - the result comes from the normal machinery from seminormal forms

Further reading I

- C. Bowman, *The many graded cellular bases of Hecke algebras*, 2017.
- C. Bowman and A. G. Cox, *Modular decomposition numbers of cyclotomic Hecke and diagrammatic Cherednik algebras: A path theoretic approach*, 2017.
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Theorem (Evseev-M.)

Suppose that $\mathcal{R}_n^\Lambda(\mathbf{Q}_I, \mathbf{K}_I)$ has a content system over \mathbb{k} and let \mathbb{K} be the field of fractions of \mathbb{k} . Then $\mathcal{R}_n^\Lambda(\mathbf{Q}_I, \mathbf{K}_I)$ is a split semisimple graded \mathbb{K} -algebra that is canonically isomorphic to a cyclotomic quiver Hecke algebra for the quiver $A_\infty \sqcup \cdots \sqcup A_\infty$ with vertex set J_ℓ .

The algebra $\mathcal{R}_n^\Lambda(\mathbf{Q}_I, \mathbf{K}_I)$ has “integral” analogues of the ψ and ψ' -bases. Unfortunately, it is not at all clear that these elements span the algebra.

Using a variation of the algebras \mathcal{W}_n^Λ we can prove:

Theorem (Evseev-M.)

Suppose that $\mathcal{R}_n^\Lambda(\mathbf{Q}_I, \mathbf{K}_I)$ has a content system over \mathbb{k} . The $\mathcal{R}_n^\Lambda(\mathbf{Q}_I, \mathbf{K}_I)$ is a graded cellular algebra

Corollary (Evseev-M.)

Let \mathcal{R}_n^Λ be a quiver Hecke algebra of type $C_e^{(1)}$. Then \mathcal{R}_n^Λ is a graded cellular algebra:ls

Further reading II

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